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The second virial coefficient of spin-$\frac{1}{2}$ interacting anyon system

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Abstract. We evaluate the propagator by the usual time-sliced manner and use it to compute the second virial coefficient of an anyon gas interacting through the repulsive potential of the form $g/r^2 (g > 0)$. All the cusps for the unpolarized spin-$\frac{1}{2}$ as well as spinless cases disappear in the $\omega \to 0$ limit, where ω is a frequency of the harmonic oscillator which is introduced as a regularization method. As g approaches zero, the result reduces to the noninteracting hard-core limit.

Since the anyon whose statistics interpolates between boson and fermion at two dimensions [1–3] was introduced, the main focus, until recently, has been on the free anyon gas, i.e. noninteraction apart from statistical interacton of the Aharonov–Bohm type. In order to investigate the statistical properties of a free anyon gas the thermodynamic quantities, such as the second virial coefficient as a function of statistical parameter α, have been calculated for both spinless [4, 5] and spin-$\frac{1}{2}$ cases [6]. The second virial coefficient of the spinless case shows the periodic dependence on α and nonanalytic behaviour at bose points. However, for the spin-$\frac{1}{2}$ case the discontinuities appear at bose points and periodicity is also removed. This difference comes from the fact that the introduction of spin allows the irregular wavefunction at origin. Even if no irregular solution is assumed in the spin-$\frac{1}{2}$ case, the cusps exist at all integer points. Recently we calculated the second virial coefficient for spinless and spin-$\frac{1}{2}$ free anyon gases [7] for various values of the self-adjoint extension [8] parameter. The result for the spin-$\frac{1}{2}$ case exhibits a completely different cusp and discontinuity structure from [6], due to the different condition for the occurrence of the irregular wavefunction at origin.

Loss and Fu [9] studied the spinless anyon gas interacting with a repulsive potential of the form g/r^2 ($g > 0$), using a similar regularization procedure to that used in [4]. They chose the $1/r^2$-potential, because it does not remove the scale invariance of theory and the path-integral solution is simply obtained. Furthermore, the probability of the overlap of two particles is always zero. This property is also valid for the spin-$\frac{1}{2}$ system with the same two-particle interaction [10]. They showed that this simple interaction makes the cusps at bose points smooth for the spinless case.

In this paper, we will compute the second virial coefficient of the spinless and spin-$\frac{1}{2}$ anyon gas interacting through this repulsive potential using the harmonic oscillator regularization. We obtain the same result with [9] for the spinless case. For the spin-$\frac{1}{2}$ case, it is found that all the cusps at both boson and fermion points of the second virial coefficient calculated under the condition that no irregular solution is assumed, become smooth. As $g \to 0$, the nonanalytic behaviour of Blum *et al* [6] is reproduced.

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We begin with the kernel for the anyon system with g/r^2 and harmonic oscillator interactions

$$K[\boldsymbol{r}_f, \boldsymbol{r}_i; T] = \int D\boldsymbol{r} \mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\, L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)} \tag{1}$$

where

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \frac{M}{2}\dot{\boldsymbol{r}}^2 - \alpha\dot{\theta} - \frac{g}{r^2} - \frac{M}{2}\omega^2\boldsymbol{r}^2 \tag{2}$$

is the Lagrangian of the system. Following a similar procedure to [11], one can obtain the Euclidean kernel as

$$\begin{aligned} G[\boldsymbol{r}_f, \boldsymbol{r}_i; \tau] &= \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\theta_f - \theta_i)} G_m[r_f, r_i; \tau] \\ G_m[r_f, r_i; \tau] &= \frac{M\omega}{2\pi \sinh \omega\tau} \exp\left[-\frac{M\omega}{2}\frac{\cosh \omega\tau}{\sinh \omega\tau}(r_i^2 + r_f^2)\right] I_{\sqrt{(m+\alpha)^2+2gM}}\left(\frac{M\omega r_i r_f}{\sinh \omega\tau}\right) \end{aligned} \tag{3}$$

where $I_\nu(x)$ is the modified Bessel function and $\tau = \mathrm{i}T$. Then we perform the Laplace transform to obtain the energy-dependent Green function:

$$\begin{aligned} \hat{G}[\boldsymbol{r}_f, \boldsymbol{r}_i; E] &= \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\theta_f - \theta_i)} \hat{G}_m[r_f, r_i; E] \\ \hat{G}_m[r_f, r_i; E] &= \frac{1}{2\pi\omega r_i r_f} \frac{\Gamma\left(\left[1 + \sqrt{(m+\alpha)^2 + 2gM} + E/\omega\right]\Big/2\right)}{\Gamma\left(1 + \sqrt{(m+\alpha)^2 + 2gM}\right)} \\ &\quad \times W_{-\frac{E}{2\omega}, \sqrt{(m+\alpha)^2+2gM}}(M\omega[\mathrm{Max}(r_i, r_f)]^2) \\ &\quad \times M_{-\frac{E}{2\omega}, \sqrt{(m+\alpha)^2+2gM}}(M\omega[\mathrm{Min}(r_i, r_f)]^2) \end{aligned} \tag{4}$$

where $W_{\kappa,\mu}(x)$ and $M_{\kappa,\mu}(x)$ are the usual Whittaker functions, and $\mathrm{Max}(x, y)$ and $\mathrm{Min}(x, y)$ are the maximum and mininum values of x and y, respectively. From the poles of the Green function, the bound state spectrum of the system is straightforwardly obtained:

$$E_{n,m} = \left(2n + 1 + \sqrt{(m+\alpha)^2 + 2gM}\right)\omega. \tag{5}$$

The plot of $E_{0,0}$ at $gM = 1$ is shown in figure 1. The cusps that happened in the absence of the $1/r^2$ potential disappear and become smooth. Therefore, by the introduction of a repulsive potential, we expect that the nonanalytic dependence on α in various thermodynamic quantities would be suppressed.

Now, we calculate the second virial coefficient B_2 of this system. The two-particle partition function Z_2 is given by

$$\begin{aligned} Z_2 &\equiv \mathrm{Tr}\exp(-\beta H_2) \\ &= 2A\lambda_T^2 Z_{rel} \end{aligned} \tag{6}$$

where H_2 is the two-particle Hamiltonian, $\beta = 1/kT$, A is the area of the system, $\lambda_T = (2\pi/kTM)^{1/2}$ is the thermal de Broglie wavelength, and Z_{rel} is the partition function in relative coordinates. The second virial coefficient then is

$$\begin{aligned} B_2(\alpha, T) &= \frac{A}{2} - 2\lambda_T^2 Z_{rel} \\ &= \frac{A}{2} - 2\lambda_T^2 \sum_{n,m} \mathrm{e}^{-\beta E_{n,m}} \end{aligned} \tag{7}$$

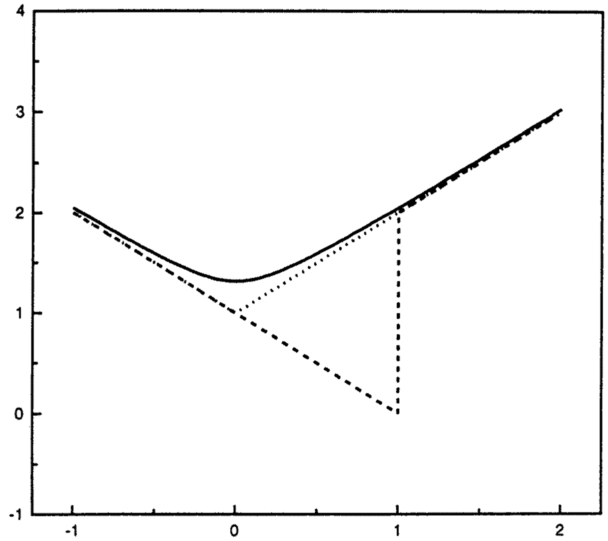

Figure 1. The bound state energy as a function of α when $n, m = 0$. Solid curve: $gM = 1$ case. Dashed line: $g = 0$ case in the presence of irregular solution. Dotted line: $g = 0$ case in the absence of irregular solution.

where $E_{n,m}$ is given in equation (5) and M is replaced by $2M$. The summation over even (odd) m's corresponds to the boson (fermion) statistics. At first, performing the summation over n, we obtain

$$B_2(\alpha, T) = \frac{A}{2} - \frac{\lambda_T^2}{\sinh \beta\omega} \sum_m \mathrm{e}^{-\beta\sqrt{(m+\alpha)^2+gM}\omega}. \tag{8}$$

Consider the spinless case by summing only over even m's. In order to regularize the infinite area, we calculate $B_2(\alpha, T) - B_2(\alpha = 0, T)$:

$$B_2(\alpha, T) - B_2(0, T) = \frac{\lambda_T^2}{\sinh \beta\omega} \sum_{m=even} [\mathrm{e}^{-\beta\sqrt{m^2+gM}\omega} - \mathrm{e}^{-\beta\sqrt{(m+\alpha)^2+gM}\omega}]. \tag{9}$$

The result in the $\omega \to 0$ limit is just that of [9] which used the different regularization procedure proposed in [4].

Next, consider the unpolarized spin-$\frac{1}{2}$ anyon case. This can be done by averaging over four possible spin states:

$$B_2(\alpha, T) - \bar{B}_2(0, T) = \frac{\lambda_T^2}{4\sinh \beta\omega} \Bigg\{ 3 \sum_{m=odd} [\mathrm{e}^{-\beta\sqrt{m^2+gM}\omega} - \mathrm{e}^{-\beta\sqrt{(m+\alpha)^2+gM}\omega}]$$
$$+ \sum_{m=even} [\mathrm{e}^{-\beta\sqrt{m^2+gM}\omega} - \mathrm{e}^{-\beta\sqrt{(m+\alpha)^2+gM}\omega}] \Bigg\} \tag{10}$$

where $\bar{B}_2(0, T)$ is the averaged $B_2(0, T)$ which cannot be determined but has no α-dependence. We show the $\omega \to 0$ limit of $B_2(\alpha, T) - \bar{B}_2(0, T)$ as a function of α for $g = 0, 0.05, 0.1$ and 1 in figure 2. When $g > 0$, the second virial coefficient has no cusps for all α as expected. As $g \to 0$, the previous result [6] is reproduced: $|\alpha| - 2\alpha^2$ for boson point and $3|\alpha| - 2\alpha^2$ for fermion point†. As a result, the repulsive interaction removes all the cusps at both boson

† Of course, the order of limits $\alpha, g \to 0$ is also crucial in this case. See the detailed discussion in [9] for this problem.

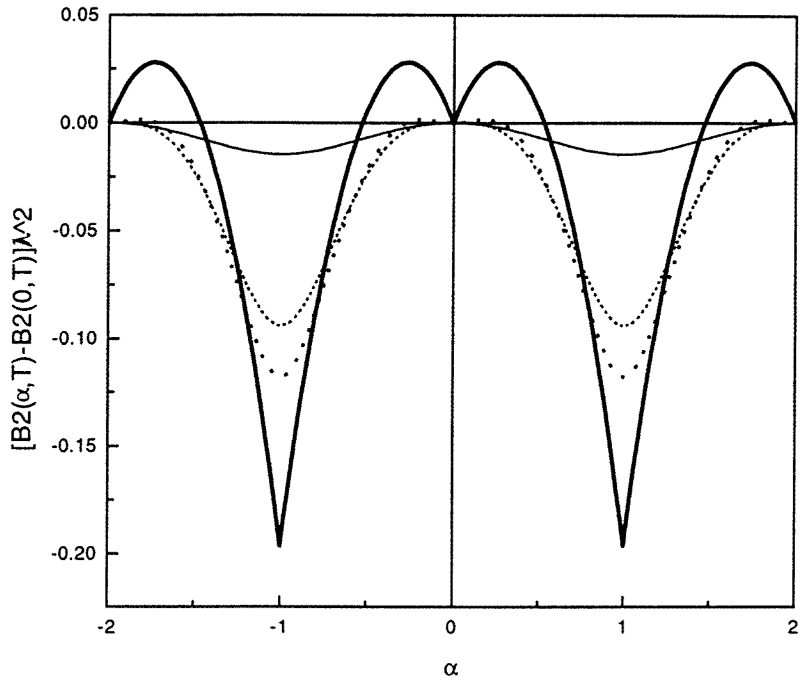

Figure 2. $[B_2(\alpha, T) - \bar{B}_2(0, T)]/\lambda_T^2$ as a function of α at various gM values. Thick solid curve: $gM = 0$. Dotted curve: $gM = 0.05$. Short-dotted curve: $gM = 0.1$. Thin solid curve: $gM = 1$.

and fermion points for spin-$\frac{1}{2}$ case. This is the extension of the spinless case whose cusps at bosonic points become smooth to the spin-$\frac{1}{2}$ anyon case.

Even though the $1/r^2$-potential is adopted to study the interacting anyons due to its simplicity, a more realistic interaction between anyons should be introduced in order to be applicable to real physical systems. If we think of anyons as the particles carrying both magnetic flux and electrical charge, the consideration of Coulomb interaction arises naturally. We have already calculated the kernel and bound states for the Aharonov–Bohm–Coulomb system incorporating the self-adjoint extension method into the Green function formalism [12]. Though the simple harmonic oscillator regularization seems to be impossible because of the difficulty in getting the path-integral solution for the Aharonov–Bohm–Coulomb plus harmonic oscillator system, the second virial coefficient may be obtained from the appropriate phase shift method in scattering theory. This problem is now under study.

In conclusion, we found the path-integral kernel for the interacting spin-$\frac{1}{2}$ anyons with repulsive potential and harmonic oscillator, and calculated the second virial coefficient using the partition function obtained by summing the harmonic oscillator bound states. For the spinless case, the cusps at bose points became smooth as in the result of [9] which used the regularization procedure by [4]. For unpolarized spin-$\frac{1}{2}$ anyons, all the cusps at both boson and fermion points disappeared. The nonanalytic behaviour with α is reproduced when $g \to 0$.

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